II. *On Periodic Disturbance of Level Arising from the Load of Neighbouring Oceanic Tides.*

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I. *Introductory.*

IN his important observations on the lunar deflection of gravity, Dr. O. HECKER has pointed out that the force acting on the pendulum at Potsdam is a larger fraction of the moon's force when it acts towards the east or west than when it acts towards the north or south. Various explanations of this anomaly have been proposed; among them one, suggested by Prof. A. E. H. LOVE,* is that a possible cause may perhaps be found in the attraction of the tide wave in the North Atlantic. Recently Prof. A. A. MICHELSON† has found a similar result in his arduous task, successfully achieved, of obtaining a continuous record of the lunar perturbation of a very long water-level at Chicago. Prof. Sir J. LARMOR kindly suggested to me a query, whether the excess-pressure of the tide in the North Atlantic would affect much the measurement of water-level at Chicago, owing to the elastic depression of the earth's surface that it would produce.

In the present paper the calculation is undertaken in order to ascertain to what extent the consideration of tilting of the ground is important for the explanation of this geodynamical discrepancy, on the assumption that the earth consists of elastic matter of uniform density and of uniform invariable elastic constants, and that the North Atlantic may be represented roughly as a circular basin and that the tide in it is uniform or else elliptic. The curvature of the earth is neglected, which is, of course, admissible for a first estimation.

II. *General Formulæ.*

The solution of the equation of equilibrium concerning an elastic body with plane boundary, otherwise extending to infinity, subject to certain boundary conditions,

* 'Some Problems of Geodynamics,' Cambridge (1911), p. 88.

† 'The Astrophysical Journal,' vol. 39 (1914), p. 105.

has been obtained by LAMÉ and CLAPEYRON* and by J. BOUSSINESQ†; the former by making use of FOURIER's theorem, and the latter by introducing several kinds of potential function. The solution for the case where the boundary condition is a normal pressure distributed symmetrically round a point on the surface and the body is free from bodily force is given by Prof. H. LAMB‡ in a form of definite integral. I have tried to solve the problem, independently of LAMB, and obtained the solution answering to any distribution of a normal pressure, from which the solution of LAMB can be derived as a special case.§

For the present purpose let us confine our attention to the simplest case in which the boundary is subjected to a symmetrically distributed normal pressure. Take the centre of the area, in undisturbed state, on which the given normal pressure is distributed, as the origin of the co-ordinates, the inward normal to the undisturbed surface as the z-axis, and denote the distance of any point from the z-axis by r. Then corresponding to the normal pressure

$$\widehat{zz} = f(r), \quad \ldots \ldots \ldots \ldots \quad (1)$$

given at the surface $z = 0$, we shall have the following expressions for the displacement components:—

$$u_r = -\frac{z}{2\mu}\int_0^\infty Z(k)\, e^{-kz} J_1(kr)\, dk + \frac{1}{2(\lambda+\mu)}\int_0^\infty Z(k)\, e^{-kz} J_1(kr)\frac{dk}{k}, \quad \ldots \ldots \quad (2)$$

$$u_z = -\frac{z}{2\mu}\int_0^\infty Z(k)\, e^{-kz} J_0(kr)\, dk - \frac{\lambda+2\mu}{2\mu(\lambda+\mu)}\int_0^\infty Z(k)\, e^{-kz} J_0(kr)\frac{dk}{k}, \quad \ldots \ldots \quad (3)$$

in which $Z(k)$ stands for

$$Z(k) = k\int_0^a f(x)\, J_0(kx)\, x\, dx, \quad \ldots \ldots \ldots \quad (4)$$

λ, μ being LAMÉ's elastic constants, $J_0(x)$, $J_1(x)$ BESSEL's functions of zeroth and first order, and a the radius of loaded area.

Now, suppose the normal pressure (1) to arise entirely from the tidal loading in the North Atlantic basin, then the tidal loading would come into play to affect the water-level measurement at Chicago in two accounts: the one is the direct attraction of the material load, the other the deformation of the ground by the pressure

* 'CRELLE's Journal,' vol. 7, p. 400 (1831).

† 'Application des Potentiels . . . . .,' Paris (1885).

‡ 'Lond. Math. Soc. Proc.,' vol. 34, p. 276 (1902).

§ The solution, with numerous examples, will be published shortly elsewhere.

produced by that load. Dr. C. CHREE,* and afterwards more completely Prof. H. NAGAOKA,† find a formula, by using the formula obtained by BOUSSINESQ, to calculate the deviation of the direction of gravity due to the attraction of a material load on the surface of the earth.

The same result can be attained, of course, from our solution. The expression for the vertical displacement at a point on the surface can be transformed into

$$(u_z)_0 = \frac{1}{2\pi} \cdot \frac{\lambda+2\mu}{2(\lambda+\mu)\mu} \int_0^{2\pi} \int_0^a \frac{f(x)}{R'} x \, dx \, d\phi$$

by making use of NEUMANN's addition theorem for BESSEL's functions, where R' stands for

$$R' = \sqrt{(r^2 - 2rx \cos\phi + x^2)}.$$

On the other hand, if we denote the attraction constant by γ, and gravity, prior to the application of the load, by g, then the gravitation-potential at a point on the unloaded surface due to the loading can be expressed by

$$V_0 = \gamma \int_0^{2\pi} \int_0^a \frac{1}{g} \cdot \frac{f(x)}{R'} x \, dx \, d\phi,$$

provided the height of the loading material is negligibly small compared with the distance of the point under consideration from any point in the loaded area.

Comparing the above two expressions, we have

$$V_0 = \frac{2\pi\gamma}{g} \cdot \frac{2\mu(\lambda+\mu)}{\lambda+2\mu} (u_z)_0.$$

Thus the direction of gravity becomes in consequence of the attraction of the loading material inclined to the vertical at the angle ψ which will be determined by

$$\tan\psi = \frac{2\pi\gamma}{g^2} \cdot \frac{2\mu(\lambda+\mu)}{\lambda+2\mu} \left(\frac{\partial u_z}{\partial r}\right)_0, \quad \ldots \ldots \ldots \quad (5)$$

while its tilting effect is expressed by

$$\tan\phi = \left(\frac{\partial u_z}{\partial r}\right)_0. \quad \ldots \ldots \ldots \quad (6)$$

The total effect of the loading will thus be

$$\phi + \psi = \left(1 + \frac{2\pi\gamma}{g^2} \cdot \frac{2\mu(\lambda+\mu)}{\lambda+2\mu}\right) \left(\frac{\partial u_z}{\partial r}\right)_0, \quad \ldots \ldots \ldots \quad (7)$$

in a close approximation.

* 'Phil. Mag.,' V., 43, p. 177 (1897).

† Tôkyô, 'Sug. Buts. Kizi,' VI., p. 208 (1912).

III. *Uniform Loading.*

Let a be the radius of the loaded circle, h the height of the material loading, which is supposed to be uniform, and ρ its density, then we have to put

$$\left.\begin{aligned} f(r) &= -gh\rho \quad \text{for} \quad r < a, \\ &= 0 \qquad\quad ,, \quad\; r > a. \end{aligned}\right\} \qquad (8)$$

On this supposition, we get

$$Z(k) = -ahg\rho J_1(ka),$$

therefore

$$u_z = \frac{ahg\rho z}{2\mu}\int_0^\infty e^{-kz} J_0(kr) J_1(ka)\, dk$$

$$+\frac{ahg\rho(\lambda+2\mu)}{2\mu(\lambda+\mu)}\int_0^\infty e^{-kz} J_0(kr) J_1(ka)\frac{dk}{k}, \qquad (9)$$

$$\left(\frac{\partial u_z}{\partial r}\right)_0 = -\frac{ahg\rho(\lambda+2\mu)}{2\mu(\lambda+\mu)}\left[\int_0^\infty e^{-kz} J_1(kr) J_1(ka)\, dk\right]_{z=0}. \qquad (10)$$

The other components of the displacement and those of stress can be expressed in similar forms. But it is unnecessary to write them down here as they are out of our present purpose.

The integrals required here cannot be evaluated in a very simple way. Some of them are closely connected to the magnetic potential due to a circular current, or to the velocity-potential and stream function of a circular vortex, and have been discussed by various authors. In his paper on the inductance of circular coils,* Prof. NAGAOKA has devised a comparatively simple method which may be applied to evaluate all the integrals needed for the calculation of the displacement and stress in the present problems. Let us follow his method and describe it here briefly.

Put

$$R = \sqrt{(a^2+r^2-2ar\cos\theta)},$$

then by NEUMANN'S addition theorem for BESSEL'S function we have

$$J_1(kr) J_1(ka) = \frac{1}{\pi}\int_0^\pi J_0(kR)\cos\theta\, d\theta,$$

$$J_0(kr) J_1(ka) = \frac{1}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R} J_1(kR)\, d\theta.$$

* 'Phil. Mag.,' VI., 6 (1903), p. 19.

Making use of these formulæ we obtain

$$\int_0^\infty e^{-kz} J_1(kr) J_1(ka)\, dk = \frac{1}{\pi}\int_0^\pi \frac{\cos\theta\, d\theta}{(R^2+z^2)^{\frac{1}{2}}},$$

$$\int_0^\infty e^{-kz} J_0(kr) J_1(ka)\, dk = \frac{1}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2}\, d\theta - \frac{z}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2(R^2+z^2)^{\frac{1}{2}}}\, d\theta,$$

$$\int_0^\infty e^{-kz} J_0(kr) J_1(ka)\frac{dk}{k} = \frac{1}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2}(R^2+z^2)^{\frac{1}{2}}\, d\theta - \frac{z}{\pi}\int_0^\pi \frac{a-r\cos\theta}{R^2}\, d\theta.$$

To find these integrals, put

$$\alpha = \left(\frac{2}{ar}\right)^{\frac{1}{3}}, \qquad \beta = \frac{a^2+r^2+z^2}{6ar}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11)$$

$$e_1 = \frac{2\beta}{\alpha}, \qquad e_2 = \frac{1-\beta}{\alpha}, \qquad e_3 = -\frac{1+\beta}{\alpha}, \quad . \quad . \quad . \quad . \quad . \quad . \quad (12)$$

so that

$$e_3 < e_2 < e_1,$$

$$e_1+e_2+e_3 = 0,$$

and change the integration variable from θ to s by

$$\cos\theta = \alpha s+\beta,$$

then

$$\int_0^\pi \frac{\cos\theta}{(R^2+z^2)^{\frac{1}{2}}}\, d\theta = \frac{\alpha^2}{2}\int_{e_3}^{e_2} \frac{(2s+e_1)\, ds}{\{4(s-e_1)(s-e_2)(s-e_3)\}^{\frac{1}{2}}}.$$

Put again

$$s = \wp(u),$$

then, since s or $\wp(u)$ is real and lies between e_3 and e_2, $s = e_3$ and $s = e_2$ correspond to $u = \omega_3$ and $u = \omega_2$ respectively, if we take $\wp'(u)$ to be positive*; where ω_1 and ω_3 denote the real and imaginary half-period and $\omega_2 = \omega_1+\omega_3$. Thus

$$\int_0^\pi \frac{\cos\theta\, d\theta}{(R^2+z^2)^{\frac{1}{2}}} = \alpha^2\int_{\omega_3}^{\omega_2} (\tfrac{1}{2}e_1+\wp(u))\, du$$

$$= \alpha^2(\tfrac{1}{2}e_1\omega_1-\eta_1). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (13)$$

For the evaluation of the other integrals, write

$$\wp(v) = \frac{a^2+r^2-2ar\beta}{2ar\alpha}. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (14)$$

* If we assume $\wp'(u)$ to be negative, then $s = e_3$ and $s = e_2$ correspond to $u = \omega_2$ and $u = 2\omega_1+\omega_3$ respectively. But the same result will, as a matter of course, be obtained after integration.

then we shall have

$$\int_0^\pi \frac{a-r\cos\theta}{R^2(R^2+z^2)^{\frac{1}{2}}}d\theta = \frac{\alpha}{2a}\omega_1 - \frac{r^2-a^2}{4a^2r}\int_{\omega_3}^{\omega_2}\frac{du}{\wp(v)-\wp(u)},$$

$$\int_0^\pi \frac{a-r\cos\theta}{R^2}(R^2+z^2)^{\frac{1}{2}}d\theta = \frac{2}{a\alpha}(e_1\omega_1+\eta_1) - \frac{r^2-a^2}{a^2r\alpha^2}\omega_1 + \frac{r^2-a^2}{a^2r\alpha^2}\int_{\omega_3}^{\omega_2}\frac{\wp(v)-e_1}{\wp(v)-\wp(u)}du.$$

Now we have

$$\int_{\omega_3}^{\omega_2}\frac{\wp'(v)\,du}{\wp(v)-\wp(u)} = \left[\log\frac{\sigma(u+v)}{\sigma(u-v)} - 2u\zeta(v)\right]_{\omega_3}^{\omega_2}$$

$$= 2v\eta_1 - 2\omega_1\zeta(v) + 2m\pi i. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (15)$$

Hence

$$\int_0^\pi \frac{a-r\cos\theta}{R^2(R^2+z^2)^{\frac{1}{2}}}d\theta = \frac{\alpha}{2a}\omega_1 - \frac{r^2-a^2}{2a^2r\wp'(v)}\{v\eta_1-\omega_1\zeta(v)+m\pi i\}, \quad . \quad . \quad . \quad (16)$$

$$\int_0^\pi \frac{a-r\cos\theta}{R^2}(R^2+z^2)^{\frac{1}{2}}d\theta = \frac{2}{a\alpha}(e_1\omega_1+\eta_1) - \frac{r^2-a^2}{a^2r\alpha^2}\omega_1$$

$$+\frac{2(r^2-a^2)}{a^2r\alpha^2}\cdot\frac{\wp(v)-e_1}{\wp'(v)}\{v\eta_1-\omega_1\zeta(v)+m\pi i\}. \quad (17)$$

The term $m\pi i$ enters because of the many-valued property of a logarithm. The actual value of m and $\wp'(v)$ will be determined by the following consideration.

From the definition of $\wp(v)$ and e_1, e_2, e_3, it follows immediately that

$$\wp(v)-e_1 = -\frac{z^2\alpha}{2ar},$$

$$\wp(v)-e_2 = \frac{(a-r)^2\alpha}{2ar},$$

$$\wp(v)-e_3 = \frac{(a+r)^2\alpha}{2ar},$$

accordingly

$$e_2 < \wp(v) < e_1.$$

The last inequality shows that the value of v must be one of the following :—

$$\left.\begin{aligned}&\text{(i.)}\ v = (2n+1)\omega_1+(2n'+\theta)\omega_3\\ &\text{(ii.)}\ v = (2n+1)\omega_1+(2n'+2-\theta)\omega_3\end{aligned}\right\}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (18)$$

where n and n' denote any integers, positive or negative, or zero, and θ a positive number less than unity. To determine the value of m in the formulæ (16) (17) for

the value of v given in (i.) of (18), observe that the integral on the left-hand side of (15) and the function $v\eta_1 - \omega_1\zeta(v)$ change their values continuously as θ varies from 0 to 1, while m remains unchanged in this variation. In the limit as $\theta \to 0$, the value of the integral is nil and

$$2v\eta_1 - 2\omega_1\zeta(v) = 2n'\pi i,$$

and therefore we have

$$\text{(i.)} \quad m = -n'.$$

Similarly for the value of v given in (ii.) of (18), proceeding to the limit $\theta \to 0$, we find

$$\text{(ii.)} \quad m = -(n'+1).$$

The value of $\wp'(v)$ will be obtained from

$$\wp'^2(v) = 4[\wp(v)-e_1].[\wp(v)-e_2].[\wp(v)-e_3].$$

In the present case we have

$$\wp'(v) = \pm i \frac{z(a^2-r^2)}{2ar}.$$

It may easily be shown that the value of $\wp'(v)$ is a positive imaginary quantity for the value of v given in (i.) of (18) and a negative imaginary quantity for (ii.).

Therefore we have to take

$$\left.\begin{aligned} \wp'(v) &= +i\left|\frac{z(a^2-r^2)}{2ar}\right|, \\ m &= -n', \\ \text{for} \quad v &= (2n+1)\,\omega_1 + (2n'+\theta)\,\omega_3\,; \end{aligned}\right\} \quad \ldots \ldots \quad (19)$$

and

$$\left.\begin{aligned} \wp'(v) &= -i\left|\frac{z(a^2-r^2)}{2ar}\right|, \\ m &= -(n'+1), \\ \text{for} \quad v &= (2n+1)\,\omega_1 + (2n'+2-\theta)\,\omega_3\,; \end{aligned}\right\} \quad \ldots \ldots \quad (20)$$

where

$$0 < \theta < 1.$$

Lastly, one more integral which can be evaluated without the knowledge of elliptic functions is

$$\left.\begin{aligned} \int_0^\pi \frac{a - r\cos\theta}{R^2}\,d\theta &= \frac{\pi}{a} \text{ for } r < a, \\ &= 0 \quad ,, \quad r > a. \end{aligned}\right\} \quad \ldots \ldots \quad (21)$$

Substituting these values of integrals in (9), we have

$$u_z = \frac{ahg\rho z}{2\mu}\left\{\begin{bmatrix}\frac{1}{a}\,(r<a)\\ \\ 0\,(r>a)\end{bmatrix} - \frac{\alpha z\omega_1}{2a\pi} + \frac{z\,(r^2-a^2)}{2a^2r\pi\wp'(v)}\,[v\eta_1-\omega_1\zeta(v)]\right\}$$

$$+\frac{ahg\rho\,(\lambda+2\mu)}{2\mu\,(\lambda+\mu)}\left\{-\begin{bmatrix}\frac{z}{a}\,(r<a)\\ \\ 0\,(r>a)\end{bmatrix}+\frac{2}{a\alpha\pi}\,(e_1\omega_1+\eta_1)-\frac{r^2-a^2}{\pi a^2ra^2}\,\omega_1\right.$$

$$\left.+\frac{2\,(r^2-a^2)}{\pi a^2ra^2}\cdot\frac{\wp(v)-e_1}{\wp'(v)}[v\eta_1-\omega_1\zeta(v)]\right\} \quad (22)$$

in which the simplest value of v, *i.e.*, $n=0$, $n'=0$ in (i.) of (18) is taken as the representative.

At the surface it reduces simply to

$$(u_z)_0 = \frac{hg\rho\,(\lambda+2\mu)}{2\mu\,(\lambda+\mu)\,\pi}\left\{\frac{2}{\alpha}\,(\eta_1+e_1\omega_1)-\frac{r^2-a^2}{ar\alpha^2}\,\omega_1\right\}, \quad (23)$$

where

$$\alpha = \left(\frac{2}{ar}\right)^{\frac{1}{2}},$$

$$e_1 = \frac{a^2+r^2}{3ar\alpha}, \qquad e_2 = -\frac{a^2+r^2-6ar}{6ar\alpha}, \qquad e_3 = -\frac{a^2+r^2+6ar}{6ar\alpha}. \quad (24)$$

Similarly for the value of $\left(\frac{\partial u_z}{\partial r}\right)_0$ we have

$$\left(\frac{\partial u_z}{\partial r}\right)_0 = -\frac{hg\rho\,(\lambda+2\mu)}{2\mu\,(\lambda+\mu)\,\pi}\cdot a\alpha^2\,(\tfrac{1}{2}e_1\omega_1-\eta_1). \quad (25)$$

For practical purposes of calculation, it will be very convenient to transform the expressions into JACOBI's q series; q being defined by

$$q = e^{i\pi\tau}, \qquad \tau = \frac{\omega_3}{\omega_1}.$$

After WEIERSTRASS, if we put

$$l = \frac{(e_1-e_3)^{\frac{1}{4}}-(e_1-e_2)^{\frac{1}{4}}}{(e_1-e_3)^{\frac{1}{4}}+(e_1-e_2)^{\frac{1}{4}}} = \frac{(r+a)^{\frac{1}{2}}-|r-a|^{\frac{1}{2}}}{(r+a)^{\frac{1}{2}}+|r-a|^{\frac{1}{2}}},$$

then q can be computated from

$$q = \frac{l}{2}+2\left(\frac{l}{2}\right)^5+15\left(\frac{l}{2}\right)^9+150\left(\frac{l}{2}\right)^{13}+O(l^{17}).$$

At a distance from the edge of the loaded circle, l is a fairly small quantity and consequently the terms after second or third may be dispensed with. The q series of the functions needed here are as follows:—

$$\eta_1+e_1\omega_1 = \frac{\pi^2}{\omega_1}\left\{\frac{1}{4}+2\sum_{n=1}^{\infty}\frac{q^{2n}}{(1+q^{2n})^2}\right\},$$

$$\omega_1 = \frac{\pi}{2}\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}\vartheta_2^{\,2}(0),$$

$$\tfrac{1}{2}e_1\omega_1-\eta_1 = \frac{1}{8\omega_1}\left\{\frac{\vartheta'''_1(0)}{\vartheta'_1(0)}-\frac{\vartheta''_2(0)}{\vartheta_2(0)}\right\};$$

$$\vartheta_2(0) = 2q^{\frac{1}{4}}(1+q^2+q^6+q^{12}+\ldots),$$

$$\vartheta'''_1(0) = -2\pi^3q^{\frac{1}{4}}(1-3^3q^2+5^3q^6-7^3q^{12}+\ldots),$$

$$\vartheta'_1(0) = 2\pi q^{\frac{1}{4}}(1-3q^2+5q^6-7q^{12}+\ldots),$$

$$\vartheta''_2(0) = -2\pi^2q^{\frac{1}{4}}(1+3^2q^2+5^2q^6+7^2q^{12}+\ldots).$$

At the point near the edge of the loaded circle, the above expansions cease to be applicable. For this case, our object will be accomplished by using the quantity q_1, instead of q, which is defined by

$$q_1 = e^{i\pi\tau_1}, \qquad \tau_1 = -\frac{\omega_1}{\omega_3} = -\frac{1}{\tau}\cdot$$

q_1 is calculated from

$$q_1 = \frac{l_1}{2}+2\left(\frac{l_1}{2}\right)^5+15\left(\frac{l_1}{2}\right)^9+150\left(\frac{l_1}{2}\right)^{13}+O(l_1^{17}),$$

in which

$$l_1 = \frac{(e_1-e_3)^{\frac{1}{4}}-(e_2-e_3)^{\frac{1}{4}}}{(e_1-e_3)^{\frac{1}{4}}+(e_2-e_3)^{\frac{1}{4}}} = \frac{(r+a)^{\frac{1}{2}}-(2(ar)^{\frac{1}{2}})^{\frac{1}{2}}}{(r+a)^{\frac{1}{2}}+(2(ar)^{\frac{1}{2}})^{\frac{1}{2}}}.$$

For example, to calculate $\left(\frac{\partial u_z}{\partial r}\right)_0$ near the edge, we proceed like this:—

By the aid of the relation

$$\frac{\vartheta'''_1(0)}{\vartheta'_1(0)} = \frac{\vartheta''_0}{\vartheta_0}+\frac{\vartheta''_2}{\vartheta_2}+\frac{\vartheta''_3}{\vartheta_3},$$

the function $\frac{1}{2}e_1\omega_1-\eta_1$ may be transformed into

$$4\pi\left(\frac{\alpha}{2}\right)^{\frac{1}{2}}(\tfrac{1}{2}e_1\omega_1-\eta_1) = \frac{1}{\vartheta_2^{\,2}(0\,|\,\tau)}\left\{\frac{\vartheta''_0(0\,|\,\tau)}{\vartheta_0(0\,|\,\tau)}+\frac{\vartheta''_3(0\,|\,\tau)}{\vartheta_3(0\,|\,\tau)}\right\}.$$

Making use of the transformation formulæ of Theta-functions it will be easily shown that

$$\frac{\vartheta''_0(0\,|\,\tau)}{\vartheta_0(0\,|\,\tau)} = 2i\pi\tau_1+\tau_1^{\,2}\frac{\vartheta''_2(0\,|\,\tau_1)}{\vartheta_2(0\,|\,\tau_1)},$$

$$\frac{\vartheta''_3(0\,|\,\tau)}{\vartheta_3(0\,|\,\tau)} = 2i\pi\tau_1+\tau_1^{\,2}\frac{\vartheta''_3(0\,|\,\tau_1)}{\vartheta_3(0\,|\,\tau_1)}$$

and

$$\vartheta_2^2(0\,|\,\tau) = -i\tau_1\vartheta_0^2(0\,|\,\tau_1),$$

consequently we have

$$\tfrac{1}{2}e_1\omega_1 - \eta_1 = \left(\frac{2}{\alpha}\right)^{\frac{1}{2}} \frac{4}{\vartheta_0^2(0\,|\,\tau_1)}\left\{-4+\frac{\log_e q_1}{\pi^2}\left[\frac{\vartheta''_2(0\,|\,\tau_1)}{\vartheta_2(0\,|\,\tau_1)}+\frac{\vartheta''_3(0\,|\,\tau_1)}{\vartheta_3(0\,|\,\tau_1)}\right]\right\}.$$

The q_1 series for the functions required here are

$$\begin{aligned}
\vartheta_0 &= 1-2q_1+2q_1^4-2q_1^9+\dots\,,\\
\vartheta''_2 &= -2\pi^2 q_1^{\frac{1}{4}}(1+3^2q_1^2+5^2q_1^6+7^2q_1^{12}+\dots),\\
\vartheta_2 &= 2q_1^{\frac{1}{4}}(1+q_1^2+q_1^6+q_1^{12}+\dots),\\
\vartheta''_3 &= -8\pi^2(q_1+4q_1^4+9q_1^9+\dots),\\
\vartheta_3 &= 1+2q_1+2q_1^4+2q_1^9+\dots\,.
\end{aligned}$$

In the accompanying diagram the courses of $(u_z)_0$ and its slope are exhibited as functions of the distance of the point of observation from the centre of the loaded

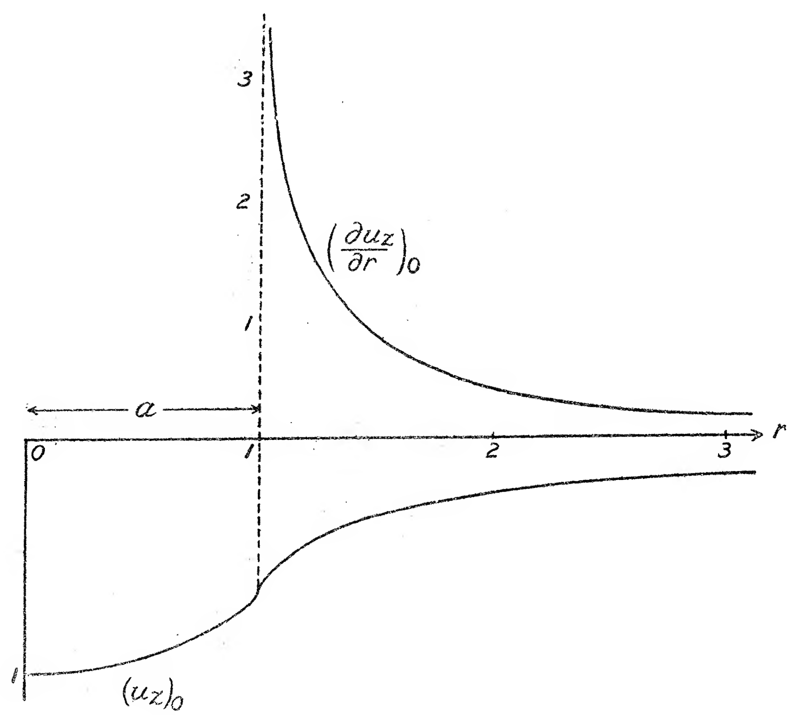

Fig. 1.

circle, with the proper scale. The lower curve thus represents the vertical displacement and the upper one the corresponding slope outside of the loaded area.

If we liken the North Atlantic to a circular basin of a large radius and determine the relative position of Potsdam or Chicago referred to the centre of it, the attraction effect and the tilting effect of the periodic filling and emptying of tide, which might

assist in producing the extra east-west force, in observations of lunar disturbance of gravity, may be computed. If we suppose the place of the observation to be not very near to the circular basin, the effect, as we see from the above diagram, is of course small, but it increases rapidly as the edge is approached.

For example, suppose the radius of the North Atlantic basin to be 2000 km., the position of Chicago to be 3000 km. from the centre, and the level of the water in this area to be raised one meter, then

$$\frac{r}{a} = 1\cdot 5, \qquad q_1 = 0\cdot 00255,$$

and

$$a\alpha^2\left(\tfrac{1}{2}e_1\omega_1 - \eta_1\right) = 0\cdot 8639.$$

Further assume that the density of sea water is 1, and in c.g.s. units,

$$\gamma = 6\cdot 65 \times 10^{-8}, \qquad g = 980,$$

$$\frac{\lambda + 2\mu}{2(\lambda+\mu)} = \tfrac{3}{4}, \qquad \mu = 6 \times 10^{11},$$

then we shall have

$$\psi = 1\cdot 17 \times 10^{-8} = 0''\cdot 0024,$$

$$\phi = 3\cdot 37 + 10^{-8} = 0''\cdot 0069.$$

Accordingly the total effect amounts to

$$\psi + \phi = 4\cdot 54 \times 10^{-8} = 0''\cdot 009.$$

It will be noticed that the effect of tilting is about three times as great as that of the attraction, so far as the material constants are assumed as above. According to Lord KELVIN,* who initiated these investigations, the direct lunar effect on the deviation of a plumbline is a maximum when the moon is at the altitude 45 degrees and amounts to 0″·017 nearly. The total effect of a tide of amplitude one metre (which is possibly two or three times the actual amount) found here is not small enough to be neglected compared with the direct effect of the moon, the former is nearly half the latter. As the tilting effect and the attraction effect of the tide wave are directly proportional to the height of the tide, the total effect oscillates in time in accordance with the law which the tide obeys. There is, in general, a difference in phase between the lunar effect and the tidal effect, which is worthy of closer investigation.† But we must bear in mind that the calculation adopted here is nothing but a rough estimation of order of magnitude, since the North Atlantic is far from circular, the tidal loading in it is never uniform.

* 'Natural Philosophy,' Part II., p. 383.

† Fortunately, the phase difference of both effects in MICHELSON'S experiment may be neglected in a rough estimation, owing to the relative position of Chicago and the centre of the North Atlantic.

Nevertheless the above analysis shows that the tidal effect on the water-level measurement, even at a point as far from the coast as Chicago, plays an important rôle and cannot be regarded as a small correction.

IV. *Elliptic Loading.*

Next, let us suppose that the tide in the North Atlantic is not uniform, but its surface is given by the equation

$$\frac{z^2}{b^2}+\frac{r^2}{a^2}=1, \quad z<0,$$

viz., the excess pressure on the bottom due to the tide diminishes on approaching the coast so as to amount to

$$\left.\begin{aligned}\widehat{zz} &= -\frac{bg\rho}{a}(a^2-r^2)^{\frac{1}{2}} \quad \text{for} \quad r<a,\\ &= 0 \qquad\qquad\qquad\quad ,, \quad\; r>a.\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (26)$$

In this case the function $Z(k)$ becomes

$$Z(k) = -abg\rho\left\{\frac{\sin ka - ka\cos ka}{k^2a^2}\right\} \quad . \quad . \quad . \quad . \quad . \quad (27)$$

Therefore we have

$$\begin{aligned}u_z &= \frac{abg\rho z}{2\mu}\int_0^\infty e^{-kz}\left\{\frac{\sin ka-ka\cos ka}{k^2a^2}\right\}J_0(kr)\,dk\\ &\quad+\frac{a^2bg\rho(\lambda+2\mu)}{2\mu(\lambda+\mu)}\int_0^\infty e^{-kz}\left\{\frac{\sin ka-ka\cos ka}{k^3a^3}\right\}J_0(kr)\,dk,\end{aligned} \quad (28)$$

and

$$\left(\frac{\partial u_z}{\partial r}\right)_0 = -\frac{abg\rho(\lambda+2\mu)}{2\mu(\lambda+\mu)}\left[\int_0^\infty e^{-kz}\left\{\frac{\sin ka-ka\cos ka}{k^2a^2}\right\}J_1(kr)\,dk\right]_{z=0} \quad . \quad . \quad (29)$$

The integrals contained in the above can be obtained by expanding the trigonometrical functions into power series of k and making use of the formula

$$\int_0^\infty e^{-kz}k^nJ_m(kr)\,dk = \frac{(n-m)!}{(r^2+z^2)^{\frac{n+1}{2}}}P_n^m\left(\frac{z}{\sqrt{(r^2+z^2)}}\right).$$

Thus

$$\begin{aligned}u_z &= \frac{abg\rho z}{\mu(r^2+z^2)^{\frac{1}{2}}}\sum_{n=1}^\infty(-1)^{n-1}\frac{n(2n-1)!}{(2n+1)!}\left(\frac{a}{\sqrt{(r^2+z^2)}}\right)^{2n-1}P_{2n-1}(\nu)\\ &\quad+\frac{a^2bg\rho(\lambda+2\mu)}{\mu(\lambda+\mu)(r^2+z^2)^{\frac{1}{2}}}\sum_{n=1}^\infty(-1)^{n-1}\frac{n(2n-2)!}{(2n+1)!}\left(\frac{a}{\sqrt{(r^2+z^2)}}\right)^{2n-2}P_{2n-2}(\nu),\end{aligned} \quad . \quad (30)$$

where

$$\nu = \frac{z}{\sqrt{(r^2+z^2)}}.$$

These series converge for $\sqrt{(z^2+r^2)} > a$, and are applicable in this region.

At the surface we have to put $z = 0$ and $\nu = 0$. Since

$$P_{2n-2}(0) = (-1)^{n-1}\frac{1.3\ldots 2n-3}{2.4\ldots 2n-2},$$

we have

$$\int_0^\infty \frac{\sin ka - ka\cos ka}{k^3a^3} J_0(kr)\,dk = \frac{1}{3r}F\left(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{5}{2}, \frac{a^2}{r^2}\right)$$

$$= \frac{1}{2a}\left\{\left(1-\frac{r^2}{2a^2}\right)\sin^{-1}\frac{a}{r} + \frac{r}{2a}\left(1-\frac{a^2}{r^2}\right)^{\frac{1}{2}}\right\}. \quad . \quad (31)$$

Consequently

$$(u_z)_0 = \frac{abg\rho(\lambda+2\mu)}{4\mu(\lambda+\mu)}\left\{\left(1-\frac{r^2}{2a^2}\right)\sin^{-1}\frac{a}{r} + \frac{r}{2a}\left(1-\frac{a^2}{r^2}\right)^{\frac{1}{2}}\right\} \quad . \quad . \quad . \quad . \quad (32)$$

for $r \geqq a$.

To find the expression for the vertical displacement within the loaded circle we proceed as follows:—

Making use of the power series of Bessel's function, we have

$$\int_0^\infty e^{-kz}\left\{\frac{\sin ka - ka\cos ka}{k^3a^3}\right\} J_0(kr)\,dk = \frac{1}{a}\sum_{n=0}^{\infty}\frac{(-1)^n}{(n\,!)^2}\left(\frac{r}{2a}\right)^{2n}\Omega_{2n}\left(\frac{z}{a}\right), \quad . \quad . \quad (33)$$

where Ω stands for

$$\Omega_m(x) = \int_0^\infty e^{-\lambda x}\left\{\frac{\sin\lambda - \lambda\cos\lambda}{\lambda^3}\right\}\lambda^m\,d\lambda.$$

The evaluation of the function $\Omega_m(x)$ can be undertaken by the aid of the formulæ

$$\int_0^\infty e^{-\lambda x}\frac{\sin\lambda}{\lambda}\,d\lambda = \tan^{-1}\frac{1}{x},$$

$$\int_0^\infty e^{-\lambda x}\cos\lambda\,d\lambda = \frac{x}{1+x^2}.$$

A little calculation will give us

$$\Omega_0(x) = \frac{\pi}{4} - \tfrac{1}{2}\left\{x + \tan^{-1}x - x^2\tan^{-1}\frac{1}{x}\right\},$$

$$\Omega_1(x) = 1 - x\tan^{-1}\frac{1}{x},$$

$$\Omega_2(x) = \tan^{-1}\frac{1}{x} - \frac{x}{1+x^2},$$

$$\Omega_3(x) = \frac{2}{(1+x^2)^2},$$

and, in general,

$$\Omega_m(x) = (-1)^{m-1} \frac{d^{m-3}}{dx^{m-3}}\left(\frac{2}{(1+x^2)^2}\right) \quad m > 2.$$

Thus the integral on the left-hand side of (33) can be expanded in an ascending power series of r/a which probably converges for limited values of r if the value of z is fixed. This series and that found in (30) have a common region in which they are both convergent and therefore they must be congruent to each other in that region. On the proof of this proposition we shall not enter, but we shall find the region of convergency of the latter series at the boundary.

Expand $\Omega_{2n}\left(\frac{z}{a}\right)$ into a power series of z/a, supposing z/a to be sufficiently small, then the first term of it will be $(-1)^n\, 2\,(n-1)\,(2n-2)\,!\,\frac{z}{a}$. If we retain only the terms which contain the first power of z/a in the series of (33), its general term will then be

$$\frac{2\,(n-1)\,(2n-2)\,!}{(n\,!)^2\, 2^{2n}}\left(\frac{r}{a}\right)^{2n}\cdot\frac{z}{a}.$$

The series which has this expression as its general term converges obviously for the values of r smaller than a. Thus the expansion (33) applies for $r < a$ when z is an infinitesimal.

Since, for $z = 0$,

$$\Omega_0(0) = \frac{\pi}{4}, \quad \Omega_2(0) = \frac{\pi}{2}, \quad \Omega_4(0) = 0, \quad \Omega_6(0) = 0, \quad \ldots,$$

we have

$$\int_0^\infty \frac{\sin ka - ka\cos ka}{k^3a^3}\, \mathrm{J}_0(kr)\, dk = \frac{\pi}{4a}\left\{1-\tfrac{1}{2}\left(\frac{r}{a}\right)^2\right\}, \quad \ldots \quad (34)$$

for $r \leqq a$.

Hence

$$(u_z)_0 = \frac{\pi abg\rho\,(\lambda+2\mu)}{8\mu\,(\lambda+\mu)}\left\{1-\tfrac{1}{2}\left(\frac{r}{a}\right)^2\right\}, \quad \ldots \quad (35)$$

for $r \leqq a$.

Quite similar arguments may be employed to find the expression for the tilting. We shall have

$$\left.\begin{aligned} \int_0^\infty \frac{\sin ka - ka\cos ka}{k^2a^2}\, \mathrm{J}_1(kr)\, dk &= \frac{r}{2a^2}\left\{\sin^{-1}\frac{a}{r} - \frac{a}{r}\left(1-\frac{a^2}{r^2}\right)^{\frac{1}{2}}\right\}, \quad r \geqq a, \\ &= \frac{\pi r}{4a^2}, \qquad r \leqq a. \end{aligned}\right\} \quad . \quad (36)$$

Consequently

$$\left(\frac{\partial u_z}{\partial r}\right)_0 = -\frac{bg\rho(\lambda+2\mu)}{4\mu(\lambda+\mu)} \cdot \frac{r}{a}\left\{\sin^{-1}\frac{a}{r} - \frac{a}{r}\left(1-\frac{a^2}{r^2}\right)^{\frac{1}{2}}\right\}, \quad . \quad . \quad . \quad . \quad (37)$$

for $r \geqq a$.

The general march of the function $(u_z)_0$ and $\left(\frac{\partial u_z}{\partial r}\right)_0$ are shown in the attached diagram (fig. 2) in a suitable scale. The lower curve represents the vertical displacement and the upper one the slope of the ground. The course of the curves is very similar to that of fig. 1, as we would expect, except at the points very near to the edge of the loaded area, where, as there is no discontinuity in the load in this case, the slope remains finite.

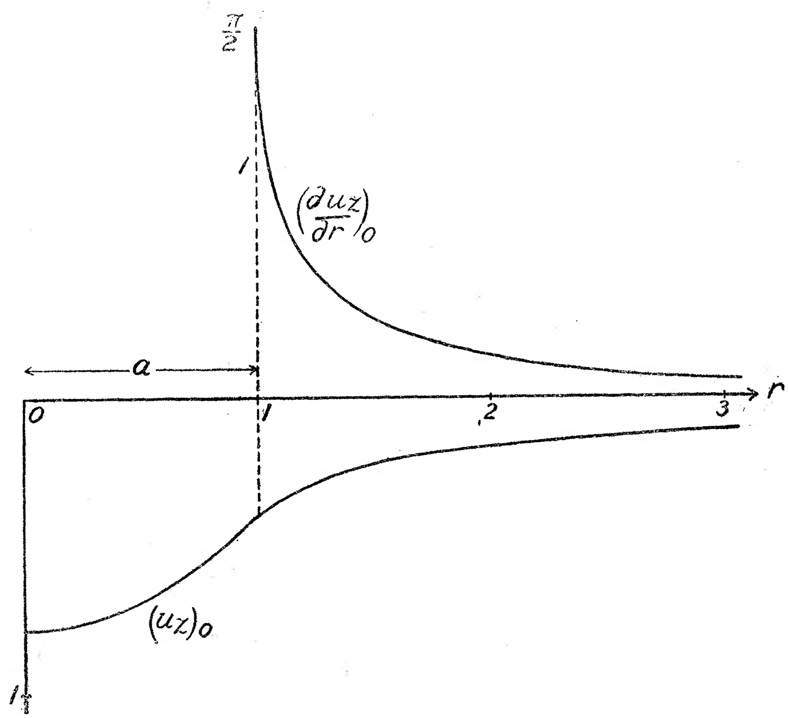

Fig. 2.

Let us take an example, with the same assumption regarding the various constants and the position of the point of observation as in the former example, and suppose that the total amount of the load is the same as before, *i.e.*, the mean height of the tide is one metre. Then we shall have

$$\psi = 1{\cdot}12 \times 10^{-8} = 0''{\cdot}0023,$$

$$\phi = 3{\cdot}21 \times 10^{-8} = 0''{\cdot}0066,$$

and

$$\psi + \phi = 4{\cdot}33 \times 10^{-8} = 0''{\cdot}009,$$

nearly the same as the result in the former example.

If we suppose the place of the observation to be nearly at the edge of the loaded area, then

$$\psi = \quad 5{\cdot}0 \times 10^{-8} = 0''{\cdot}01,$$

$$\phi = 14{\cdot}4 + 10^{-8} = 0''{\cdot}03,$$

and

$$\psi + \phi = 0''{\cdot}04,$$

amounting to more than the direct lunar effect itself, notwithstanding that the load falls off towards the coast.